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TRANSIENT HEAT AND MASS TRANSFER IN FULLY DEVELOPED LAMINAR TUBE FLOWS

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Abstract—The problem of transient heat or mass transfer in fully developed laminar tube flows has been solved exactly. The solutions for the impulse and step function response are considered separately for the boundary conditions for a constant wall temperature or concentration and for an adiabatic wall. The solution permits calculation of the local temperatures or concentrations as well as heat or mass fluxes. Typical results are presented for downstream conditions of local temperature and Nusselt number.

NOMENCLATURE

a, internal radius of the tube;
a_{jk}, matrix element defined by equations (16) and (17);
b_{jk}, matrix element defined by equation (14);
C, mass concentration;
C_0, characteristic mass concentration;
C_v, specific heat;
$\mathcal{C}$, $= C/C_0$, normalized mass concentration;
D, coefficient of mass diffusion;
d_n, D_n, parameters defined in equation (24);
E_1, parameter defined in equation (38);
F_n, matrix element defined in equation (12);
G_n, normalizing factor defined in equation (7);
g_{kj}, parameter defined in equation (23);
h_n, parameter defined in equation (24);
J_0, J_1, Bessel functions of the first and second order;
k, coefficient of thermal conductivity;
$\bar{k}$, $= \sqrt{(\omega/v)}$, a frequency parameter;
Nu, $= -\left(\frac{\partial\theta}{\partial R}\right)_{R=1}$, Nusselt number;
p_0, average pressure gradient;
Pe, $= \rho a C_v v_0/k$, Péclét number for heat transfer;
Pe', $= a v_0/D$, Péclét number for mass transfer;
Pr, $= \mu C_v/k$, Prandtl number;
r, radial distance;
R, $= r/a$, normalized radial distance;
Re, $= \rho a v_0/\mu$, Reynolds number;
Re, real value;
t, time;
T_1, T_2, characteristic initial temperature upstream and downstream respectively;
U, step function input;
v, axial velocity;
v_0, average axial velocity;
V, v/v_0, normalized axial velocity;
x, axial distance;
X, $= x/a$, normalized axial distance;
α, eigenvalue as given by equations (8), (9) or (10);
β, eigenvalue as given by equations (6) or (21);
δ, impulse function input;
δ_{jk}, Kronecker delta function defined by equation (15);
ζ_n, parameter defined by equation (18);
θ, $= (T_1 - T)/(T_1 - T_2)$, normalized temperature;

θ_m, $= \int_0^1 \theta 2R\, dR$, cross sectional average temperature;
θ_M, $= \int_0^1 V\theta 2R\, dR$, convective average temperature;
λ, eigenvalue as given by equations (18) or (19);
μ, dynamic viscosity;
ν, kinematic viscosity;
ξ, dummy variable;
ρ, density;
τ, $= tv_0/a$, normalized time;
ϕ_n, $= \int_0^1 \frac{J_0(\alpha_n R)R^3\, dT}{G_n}$, a parameter;
ω, frequency.

INTRODUCTION

INTEREST in the problem of heat transfer in tube flow dates back over a hundred years. Graetz [1] presented a solution to the problem of steady heat transfer to a constant temperature tube wall in the thermal entrance region from a Poiseuille flow in 1885. Since that date experimental and theoretical contributions to the steady state heat transfer literature have been numerous. Analytical solutions for fully developed laminar flows have been presented by Siegel *et al.* [2], Singh [3], Schenk and Han [4] and Hwang *et al.* [5] to mention only a few. Taylor [6], in 1953, appears to have been the first to experimentally and theoretically study the problem of transient convective-diffusion of mass in a tube. Millsaps and Pohlhausen [7] presented an analytical solution of transient heat transfer in a Poiseuille tube flow with no axial diffusion of heat. Since Taylor's original contribution [6], Aris [8], Philip[9], Gill [10] and Lighthill [11] have presented approximate analytical solutions to the problem of mass dispersion in fully developed laminar pipe flow. These authors have presented analytical predictions for mean concentrations with adiabatic tube walls; hence their results are of limited use in heat transfer problems. In addition, all of these authors used some restricting assumptions such as large Péclét number, large values of time or no axial diffusion. More recently Tseng and Besant [12] and Gill and Sankarasubramanian [13] have presented exact solutions for the problem of mass dispersion in a tube with adiabatic walls for a step function input. The solution of Tseng and Besant [12] is a general solution of this problem which is readily extended to similar problems, whereas Gill and Sankarasubramanian [13] have presented a solution for the mean concentration. A numerical analysis of the problem of transient mass dispersion in a tube has been presented by Ananthakrishnan *et al.* [14]. Their numerical results have only been presented for a few cases.

In this paper the problem of unsteady heat or mass flux in a Poiseuille tube flow is analyzed for incompressible flows with constant properties. The step function and impulse function response solutions are presented for the cases of adiabatic and constant temperature walls. The effects of unsteady flows are discussed. No heat dissipation is considered in the heat convection equation nor is the generation of mass species considered in the convective-diffusion equation. The results are valid for heat or mass transfer, but not simultaneous heat and mass transfer unless the coupling effects are negligible. Some typical graphical results are presented from the analytical results. However, any attempt at completeness for such a graphical presentation will be subject to the same limitations that exist for three dimensional or unsteady boundary layers; that is three independent variables and one parameter. In this case, time, axial length and radius are the independent coordinates and the Péclét number is an independent parameter. Consequently, the analytical solutions themselves are the most practical form of presentation.

PROBLEM FORMATION

The equation of convective heat transfer in a fluid with constant properties for fully developed tube flow can be expressed as:

$$\rho C_v\left[\frac{\partial T}{\partial t} + v\frac{\partial T}{\partial x}\right] = k\left[\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial x^2}\right].$$

This equation may be written in the nondimensional form:

$$\frac{\partial \theta}{\partial \tau} + V\frac{\partial \theta}{\partial X} = \frac{1}{Pe}\left[\frac{\partial^2 \theta}{\partial R^2} + \frac{1}{R}\frac{\partial \theta}{\partial R} + \frac{\partial^2 \theta}{\partial X^2}\right] \quad (1)$$

where the Péclét number, Pe, is equal to the product of the Prandtl number, Pr, and the Reynolds number, Re.

The equation governing the mass dispersal of solute in a laminar pipe flow is

$$\frac{\partial C}{\partial t} + V\frac{\partial C}{\partial x} = D\left[\frac{\partial^2 C}{\partial r^2} + \frac{1}{r}\frac{\partial C}{\partial r} + \frac{\partial^2 C}{\partial x^2}\right].$$

The above equation may be transformed into the nondimensional form as

$$\frac{\partial C}{\partial \tau} + V\frac{\partial C}{\partial X} = \frac{1}{Pe'}\left[\frac{\partial^2 C}{\partial R^2} + \frac{1}{R}\frac{\partial C}{\partial R} + \frac{\partial^2 C}{\partial X^2}\right].$$

The heat transfer case will be discussed only, since this equation is the same as for heat transfer.

In heat transfer a number of boundary conditions may have physical significance. For transient heat transfer in tube flow these are: the constant wall temperature,

$$\theta(X, R = 1, \tau) = 0 \qquad X > 0 \quad (2)$$

in adiabatic wall,

$$\frac{\partial \theta}{\partial R}(X, R = 1, \tau) = 0 \quad (3)$$

and the general linear wall condition

$$\theta(X, R = 1, \tau) + b\frac{\partial \theta}{\partial R}(X, R = 1, \tau) = 0, X > 0 \quad (4)$$

where b is a constant. The initial conditions are taken to be

$$\theta(X, R, \tau = 0) = \begin{cases} U \text{ for the step input} \\ \delta \text{ for the impulse input} \end{cases} \quad (5)$$

where $\theta(X, R, \tau)$ is finite for $-\infty < X < \infty$, $0 \leqslant R \leqslant 1$ and $0 < \tau < \infty$.

Solution. Equation (1) has a particular solution

$$\theta = \sum_{n=1}^{N} F_n \frac{J_0(\alpha_n R)}{G_n} \exp\left(-\beta^2\tau/Pe - \lambda\tau + i\beta X\right) \quad (6)$$

where the normalizing factor G_n is given by

$$G_n = \left[\int_0^1 J_0^2(\alpha_n R)\, R\, \mathrm{d}R\right]^{\frac{1}{2}} \quad (7)$$

and α_n is determined by the boundary conditions; for the constant temperature wall,

$$J_0(\alpha_n) = 0 \quad (8)$$

for the adiabatic wall,

$$J_1(\alpha_n) = 0 \quad (9)$$

for the general linearized wall condition

$$J_0(\alpha_n) - b\alpha_n J_1(\alpha_n) = 0. \quad (10)$$

The term F_n is found by substituting equation (6) into equation (1) giving

$$-\lambda \sum_{n=1}^{N} F_n \frac{J_0(\alpha_n R)}{G_n} + i\beta V(R) \sum_{n=1}^{N} F_n \frac{J_0(\alpha_n R)}{G_n} = \frac{1}{Pe}\sum_{n=1}^{N} F_n(-\alpha_n^2)\frac{J_0(\alpha_n R)}{G_n}. \quad (11)$$

This equation may be written in matrix form by first operating on equation (11) with

$$\int_0^1 \frac{J_0(\alpha_j R)}{G_j} R\, \mathrm{d}R \qquad j = 1, 2, 3 \ldots N$$

to give

$$\begin{bmatrix} \lambda & & & & & \\ & \lambda & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \lambda \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ \cdot \\ \cdot \\ \cdot \\ F_n \end{bmatrix} \begin{bmatrix} i\beta b_{11} + \frac{\alpha_1^2}{Pe} & i\beta b_{12} & & i\beta b_{1N} \\ i\beta b_{21} & i\beta b_{22} + \frac{\alpha_2^2}{Pe} & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ i\beta b_{N1} & \cdot & \cdot & i\beta b_{NN} + \frac{\alpha_N^2}{Pe} \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ \cdot \\ \cdot \\ \cdot \\ F_N \end{bmatrix} \quad (12)$$

or

$$[A - I\lambda][F] = 0 \quad (13)$$

where

$$b_{jK} = \int_0^1 V(R) \frac{J_0(\alpha_j R)\, J_0(\alpha_k R)}{G_j G_K} R\, \mathrm{d}R, \quad (14)$$

$$\delta_{jK} = \int_0^1 \frac{J_0(\alpha_j R)\, J_0(\alpha_k R)}{G_j G_K} R\, \mathrm{d}R, \quad (15)$$

and matrix A has the elements

$$a_{jk} = i\beta b_{jk} + \alpha_j^2 \delta_{jk}/Pe\, j, k = 1, 2, \ldots, N. \quad (16)$$

Two limiting cases are for Péclét number large

$$a_{jk} = i\beta b_{jk}\, j, k = 1, 2, 3, \ldots, N \quad (17)$$

and for Péclét number small a_{jk} takes the same form as equation (16) where, $|a_{kk}| \gg |a_{jk}|$ for $j \neq k$.

The eigenvalues of A can be readily found by computer since $a_{jk} = a_{kj}$. For Pe large the eigenvalues $\lambda_n, n = 1, 2, \ldots, N$ will have the simple form.

$$\lambda_n = i\beta\zeta_n \text{ where } \zeta_n \text{ is real} \quad (18)$$

and for Pe small the diagonal terms are dominant and the eigenvalues were found by using a method suggested by Faddeev and Faddeeva [15] which have the approximate form

$$\lambda_n = i\beta b_{nn} + \frac{\alpha_n^2}{Pe} + \beta^2 Pe \sum_{\substack{j=1 \\ j\neq n}}^{N} \frac{b_j^2 n}{\alpha_j^2 - \alpha_n^2} \quad (19)$$

and associated mth component of nth eigenvector g_{mn} are $n, m = 1, 2 \ldots N$

$$\begin{bmatrix} 1 & Pe\frac{a_{12}}{\alpha_1^2 - \alpha_2^2} & \cdots & Pe\frac{a_{1N}}{\alpha_1^2 - \alpha_N^2} \\ Pe\frac{a_{21}}{\alpha_2^2 - \alpha_1^2} & 1 & \cdots & Pe\frac{a_{2N}}{\alpha_2^2 - \alpha_N^2} \\ \cdots & \cdots & & \cdots \\ Pe\frac{a_{N1}}{\alpha_N^2 - \alpha_1^2} & Pe\frac{a_{N2}}{\alpha_N^2 - \alpha_2^2} & \cdots & 1 \end{bmatrix} \quad (20)$$

The general solution of equation (1) now becomes

$$\theta = \int_{-\infty}^{\infty} \sum_{j=1}^{N} \sum_{n=1}^{N} h_j g_{nj} \frac{J_0(\alpha_n R)}{G_n} \exp\left[-\frac{\beta^2}{Pe}\tau - \lambda_j \tau + i\beta X\right] \mathrm{d}\beta \quad (21)$$

where h_j is determined by the initial conditions

$$\theta(X, R, 0) = \int_{-\infty}^{\infty} \sum_{j=1}^{N} \sum_{n=1}^{N} h_j g_{nj} \frac{J_0(\alpha_n R)}{G_n} \times e^{i\beta X}\, \mathrm{d}\beta. \quad (22)$$

Using the Fourier transform and operating with respect to

$$\int_0^1 \frac{J_0(\alpha_K R)}{G^K} R\,\mathrm{d}R \text{ gives}$$

$$\frac{1}{2\pi}\int_0^1 \int_{-\infty}^{\infty} \theta(\xi, R, 0)\,\mathrm{e}^{-i\beta\xi}\,\mathrm{d}\xi \frac{J_0(\alpha_k R)}{G_K} R\,\mathrm{d}R$$

$$= \sum_{j=1}^{N} h_j g_{kj} \qquad k = 1, 2, \ldots, N. \tag{23}$$

The initial conditions as presented in equation (5) gives for large Péclét number

$$h_n = \frac{1}{2\pi} D_n \int_{-\infty}^{\infty} \theta(\xi, R, 0)\,\mathrm{e}^{-i\beta\xi}\,\mathrm{d}\xi \tag{24}$$

where

$$d_j = \int_0^1 \frac{J_0(\alpha_j R)}{G_j} R\,\mathrm{d}R \quad \text{and} \quad D_n = \sum_{j=1}^{N} d_j g_{jn}.$$

For large values of Péclét number the eigenvalues λ_n can be replaced by $i\beta\zeta_n$ as in equation (18). This gives the solution of equation (1) to be

$$\theta(X, R, \tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sum_{j=1}^{N}\sum_{n=1}^{N} D_j \theta(\xi, R, 0)$$

$$\mathrm{e}^{-i\beta\xi}\,\mathrm{d}\xi\, g_{nj}\frac{J_0(\alpha_n R)}{Gn}$$

$$\exp\left(-\frac{\beta^2\tau}{Pe} + i\beta X - i\beta\zeta_j\tau\right)\mathrm{d}\beta.$$

If $\theta(\xi, R, 0) = U(\xi)$, a step input, then

$$\theta(X, R, \tau) = \sum_{j=1}^{N}\sum_{n=1}^{N} D_j g_{nj}\frac{J_0(\alpha_n R)}{G_n}\frac{1}{2}\,\mathrm{erfc}$$

$$\times\left[\frac{X - \zeta_j\tau}{(4\tau/Pe)^{\frac{1}{2}}}\right]. \tag{25}$$

If $\theta(\xi, R, 0) = \delta(\xi)$, an impulse, then

$$\theta(X, R, \tau) = \frac{1}{\sqrt{\pi}}\sum_{j=1}^{N}\sum_{n=1}^{N} D_j g_{nj}\frac{J_0(\alpha_n R)}{\sqrt{(4\tau/Pe\,.\,G_n)}}$$

$$\exp\left[-\frac{X - \zeta_j\tau)^2}{4\tau/Pe}\right]. \tag{26}$$

The mean cross sectional temperature is given by

$$\theta_m = 2\frac{1}{\sqrt{\pi}}\sum_{j=1}^{N}\sum_{n=1}^{N} D_j g_{nj}\frac{d_n}{\sqrt{(4\tau/Pe)}}$$

$$\exp\left[-\frac{(X - \zeta_j\tau)^2}{4\tau/Pe}\right] \tag{27}$$

for the impulse function and by

$$\theta_m = \sum_{j=1}^{N}\sum_{n=1}^{N} D_j g_{nj} d_n\,\mathrm{erfc}\left[\frac{X - \zeta_j\tau}{(4\tau/Pe)^{\frac{1}{2}}}\right] \tag{28}$$

for the step function.

The convective average temperature is given by

$$\theta_M = 2\frac{1}{\sqrt{\pi}}\sum_{j=1}^{N}\sum_{n=1}^{N} D_j g_{nj}\frac{(d_n - \phi_n)}{\sqrt{(4\tau/Pe)}}$$

$$\exp\left[-\frac{(X - \zeta_j\tau)^2}{4\tau/Pe}\right] \tag{29}$$

and by

$$\theta_M = \sum_{j=1}^{N}\sum_{n=1}^{N} D_j g_{nj}(d_n - \phi_n)\frac{1}{2}$$

$$\times\,\mathrm{erfc}\left[\frac{X - \zeta_j\tau}{(4\tau/Pe)^{\frac{1}{2}}}\right] \tag{30}$$

for the impulse and step functions respectively.

For small values of Péclét number of eigenvalue λ_n take the form of equation (19)

$$\lambda_n = i\beta b_{nn} + \frac{\alpha_n^2}{Pe} + \beta^2 Pe\,E_n$$

where

$$E_n = \sum_{\substack{j=1\\ j\neq n}}^{N} \frac{b_{jn}^2}{\alpha_j^2 - \alpha_n^2}.$$

The approximate value of h_n is

$$h_n = \frac{1}{2\pi} d_n \int_{-\infty}^{\infty} \theta(\xi, R, 0)\, e^{-i\beta\xi}\, d\xi.$$

The solution of equation (1) becomes

$$\theta(X, R, \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{j=1}^{N} \sum_{\substack{n=1\\ n\neq j}}^{N} d_j \frac{Pe\, a_{nj}}{\alpha_n^2 - \alpha_j^2} \times \theta(\xi, R, 0)\, e^{-i\beta\xi} \frac{J_0(\alpha_n R)}{G_n} \times \exp\left[-\frac{\beta^2}{Pe} + i\beta X - (i\beta b_{jj} + \alpha_j^2/Pe + \beta^2 Pe\, E_j)\tau \right] d\beta\, d\xi$$
$$+ \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{j=1}^{N} d_j\, \theta(\xi, R, 0)\, e^{-i\beta\xi}\, d\xi \times \frac{J_0(\alpha_j R)}{G_j} \exp\left[-\beta^2\tau/Pe + i\beta X - (i\beta b_{jj} + \alpha_j^2/Pe + \beta^2 Pe\, E_j)\tau \right] d\beta\, d\xi.$$

If $\theta(\xi, R, 0) = U(\xi)$, the step function, we get

$$\theta(X, R, \tau) = \sum_{j=1}^{N} d_j e^{-\alpha_j^2\tau/Pe} \frac{J_0(\alpha_j R)}{G_n} \frac{1}{2} \operatorname{erfc}\left[\frac{X - b_{jj}\tau}{[4\tau(1/Pe + Pe\, E_j)]^{\frac{1}{2}}} \right]$$
$$+ \frac{1}{2\sqrt{\pi}} \sum_{j=1}^{N} \sum_{\substack{n=1\\ n\neq j}}^{N} d_j \frac{Pe\, b_{nj}}{\alpha_n^2 - \alpha_j^2} \frac{J_0(\alpha_n R)}{G_n} \frac{\exp\{-\alpha_n^2\tau/Pe - (X - b_{jj}\tau)^2/[4\tau(1/Pe + Pe\, E_j)]\}}{[\tau(1/Pe + Pe\, E_j)]^{\frac{1}{2}}}. \tag{31}$$

If $\theta(\xi, R, 0) = \delta(\xi)$, the impulse function, then

$$\theta(X, R, \tau) = \frac{1}{\sqrt{\pi}} \sum_{j=1}^{N} d_j e^{-\alpha_j^2\tau/Pe} \cdot \frac{J_0(\alpha_j R)}{G_j} \frac{1}{\sqrt{[4\tau(1/Pe + Pe\, E_j)]}}$$
$$\exp[-(X - b_{jj}\tau)^2/4\tau(1/Pe + Pe\, E_j)] + \frac{1}{2\sqrt{\pi}} \sum_{j=1}^{N} \sum_{\substack{n=1\\ n\neq j}}^{N} d_j \frac{Pe\, b_{nj}}{\alpha_n^2 - \alpha_j^2} \frac{J_0(\alpha_n R)}{G_n}$$
$$\cdot e^{-\alpha_n^2\tau/Pe} \frac{(X - b_{jj}\tau) \exp\{-(X - b_{jj}\tau)^2/[4\tau(1/Pe + Pe\, E_j)]\}}{2[\tau(1/Pe + Pe\, E_j)]^{\frac{3}{2}}}. \tag{32}$$

The mean cross sectional temperature is given by

$$\theta_m(X,\tau) = \frac{2}{\sqrt{\pi}} \sum_{j=1}^{N} d_j^2 \, e^{-\alpha_j^2/Pe} \frac{1}{\sqrt{[4\tau(1/Pe + Pe\,E_j)]}} \exp[-(X - b_{jj}\tau)^2/4\tau(1/Pe + Pe\,E_j)]$$

$$+ \frac{1}{\sqrt{\pi}} \sum_{j=1}^{N} \sum_{\substack{n=1 \\ n \neq j}}^{N} d_j^2 \frac{Pe\, b_{nj}}{\alpha_n^2 - \alpha_j^2} e^{-\alpha_n^2/Pe} . \frac{(X - b_{jj}\tau)\exp\{-(X - b_{jj}\tau)^2/[4\tau(1/Pe + Pe\,E_j)]\}}{2[\tau(1/Pe + Pe\,E_j)]^{3/2}} \quad (33)$$

for the impulse function and by

$$\theta_m(X,\tau) = \sum_{j=1}^{N} d_j^2 \, e^{-\alpha_j^2\tau/Pe} \operatorname{erfc}\left[\frac{X - b_{jj}\tau}{[4\tau(1/Pe + Pe\,E_j)]^{1/2}}\right]$$

$$+ \frac{1}{\sqrt{\pi}} \sum_{j=1}^{N} \sum_{\substack{n=1 \\ n \neq j}}^{N} d_j^2 \frac{Pe\, b_{nj}}{\alpha_n^2 - \alpha_j^2} \frac{e^{-\alpha_n^2\tau/Pe} \exp[-(X - b_{jj}\tau)^2/4\tau(1/Pe + E_j\,Pe)]}{[\tau(1/Pe + E_j\,Pe)]^{1/2}}. \quad (34)$$

The convective average temperature is given by

$$\theta_M(X,\tau) = \frac{2}{\sqrt{\pi}} \sum d_j(d_j - \phi_j) \, e^{-\alpha_j^2/Pe} \frac{1}{\sqrt{[4\tau(1/Pe + Pe\,E_j)]}} \exp[-(X - b_{jj}\tau)^2/4\tau(1/Pe + Pe\,E_j)]$$

$$+ \frac{1}{\sqrt{\pi}} \sum_{j=1}^{N} \sum_{\substack{n=1 \\ n \neq j}}^{N} d_j(d_j - \phi_j) \frac{Pe b_{nj}}{\alpha_n^2 - \alpha_j^2} e^{-\alpha_n^2\tau/Pe} . \frac{(X - b_{jj}\tau)\exp\{-(X - b_{jj}\tau)^2/[4\tau(1/Pe + PeE_j)]\}}{2[\tau(1/Pe + Pe\,E_j)]^{3/2}} \quad (35)$$

and by

$$\theta_M(X,\tau) = \sum_{j=1}^{N} d_j(d_j - \phi_j) \, e^{-\alpha_j^2\tau/Pe} \operatorname{erfc}\left[\frac{X - b_{jj}\tau}{[4\tau(1/Pe + Pe\,E_j)]^{1/2}}\right]$$

$$+ \frac{1}{\sqrt{\pi}} \sum_{j=1}^{N} \sum_{\substack{n=1 \\ n \neq j}}^{N} d_j(d_j - \phi_j) \frac{Pe\, b_{nj}}{\alpha_n^2 - \alpha_j^2} \frac{e^{-\alpha_n^2\tau/Pe} \exp[-(X - b_{jj}\tau)^2/4\tau(1/Pe + E_j\,Pe)]}{[\tau(1/Pe + E_j\,Pe)]^{1/2}} \quad (36)$$

for the impulse and step function responses.

PULSATING FLOW CASE

The velocity profile in the pulsating pipe flow was given by Uchida [16] as

$$V = \mathrm{Re}\left\{-\frac{1}{8}\frac{iP_0}{\omega}\left[1 - \frac{J_0(\bar{k}aRi^{\frac{3}{2}})}{J_0(\bar{k}ai^{\frac{3}{2}})}\right]e^{i\omega\tau}\right\}$$

$$a_{1j} = \mathrm{Re}\,\{e^{i\omega\tau}(x_j + iy_j)\}$$

then

$$x_j = \mathrm{Re}\left\{\mathrm{sign}\,[J_0(\alpha_j)]\cdot\frac{i}{[(\bar{k}a)(\alpha_j^2 + (\bar{k}a)^2 i]}\cdot\frac{J_1(\bar{k}ai^{\frac{3}{2}})}{J_0(\bar{k}ai^{\frac{3}{2}})}\right\}$$

$$y_j = \mathrm{Im}\left\{\mathrm{sign}\,[J_0(\alpha_j)]\cdot\frac{i}{[(\bar{k}a)(\alpha_j^2 + (\bar{k}a)^2 i]}\cdot\frac{J_1(\bar{k}ai^{\frac{3}{2}})}{J_0(\bar{k}ai^{\frac{3}{2}})}\right\}.$$

The average of one complete period the value of E_1 is

$$E_1 = \frac{\omega}{2\pi}\int_0^{2\pi/\omega}\sum_{n=2}^{N}\frac{a_{1n}^2}{\alpha_n^2 - \alpha_1^2}\,d\tau$$

$$= \frac{1}{2}\sum_{n=1}^{N}\frac{1}{\alpha_n^2 - \alpha_1^2}(x_n^2 + y_n^2).$$

The equivalent diffusion coefficient $K_{\text{pulsation}}$ in the pulsating pipe flow is

$$K_{\text{pulsation}} = \frac{k}{\rho C_v} + \frac{\rho C_v}{k}\cdot 4V_m^2 a^2 E_1$$

$$= \frac{k}{\rho C_v} + \frac{a^2 V_m^2}{k/\rho C_v}\sum_{n=2}^{N}\frac{2}{\alpha_n^2 - \alpha_1^2}(x_n^2 + y_n^2).$$

DISCUSSION OF RESULTS

The solution of the problem of transient heat or mass transfer in a fully developed laminar tube flow has been presented. The results may be computed to an accuracy of about 1/10000 by considering the first ten eigen-values.

The relative influence of diffusion and convection is established by the magnitude of the Péclét number. For large values of Péclét number, which is the case for most heat transfer situations, diffusion is relatively unimportant except near the walls where the velocity is negligible. On the other hand, for the slow flow of a liquid metal or a dilute solute at a low

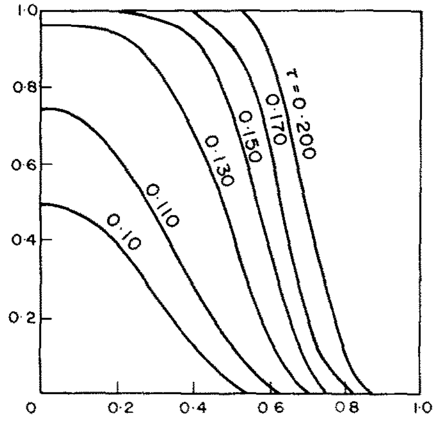

FIG. 1. Radial temperature distributions as a function of dimensionless time τ at the dimensionless position $X = 0{\cdot}1$ for a Péclét number Pe of 1000, a constant wall temperature $\theta = 0$ and the step function initial temperature distribution.

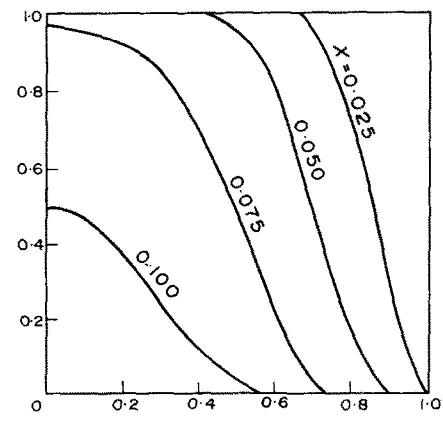

FIG. 2. Radial temperatures as a function of dimensionless position X at the dimensionless time $\tau = 0{\cdot}1$ for a Péclét number Pe of 1000, and a constant wall temperature $\theta = 0$ and the step function initial temperature distribution.

Péclét number, molecular diffusion is more important than convection. Singh [3] showed in his exact analysis of the steady state heat transfer in a tube that neglecting axial diffusion in flows with Péclét number greater than 100 gave rise to a negligible error. This same conclusion has been made by Schneider [18] and by Hsu [19] for flow in the entrance of a pipe.

The solution expressed in equations (25)–(36) may also represent the mass transfer if the temperature is replaced for the concentration and the Péclét number by its equivalent mass transfer Péclét number. Equations (26), (27), (29), (32) and (33) for the response of the impulse function, are excluded for the case of $\tau = 0$, because the magnitude of the impulse function $\delta(\tau)$ is undefined at $\tau = 0$. The Nusselt number, $Nu = -\partial\theta/\partial R|_{R=1}$, is readily computed from equations (25, (26) or (29), (30) depending on the Péclét number and the initial conditions. A typical plot of Nusselt number versus time for step function response is shown in Fig. 3. The

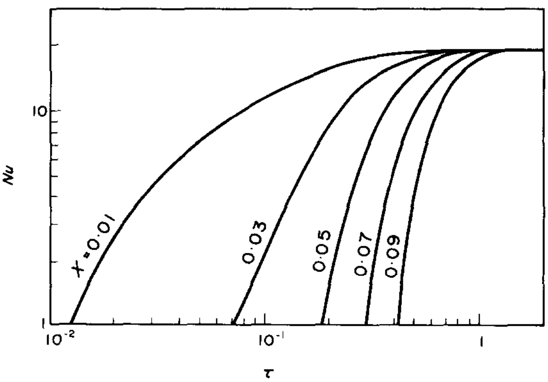

FIG. 3. Nusselt number Nu at the wall vs. dimensionless time τ at several dimensionless positions downstream for Péclét number Pe of 1000 and the step function initial temperature distribution.

Nusselt number asymptotically approaches the steady state values found near $X = 0$. For large distances downstream the asymptotic steady state value of Nusselt number is 3·66 for $Pe = \infty$. With the initial distribution of step function form the typical temperature profiles for the case of constant temperature wall are shown in Fig. 1 and 2 for the case of adiabatic walls are shown in Fig. 4 and 5. Figures 6 and 7 use the particular temperature at, $X = 0{\cdot}0001$, $\tau = 0{\cdot}0001$, $R = 0$, as a reference level to show the variation of temperature profile with the adiabatic wall and impulse form of the initial conditions. Figures 8 and 9 present the same results for the case of constant wall temperature. Mean temperature changes are given in Fig. 10

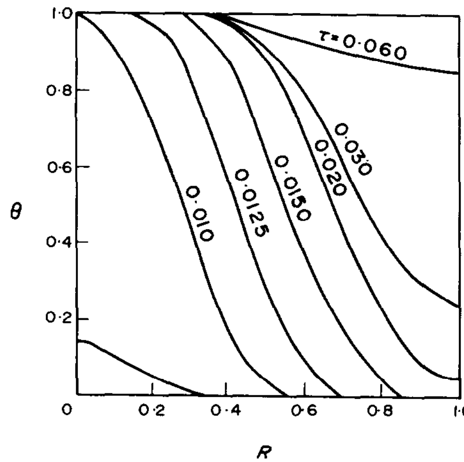

FIG. 4. Radial temperature distributions as a function of dimensionless time τ at a dimensionless position $X = 0{\cdot}1$ for a Péclét number Pe of 1000, an adiabatic wall and the step function initial temperature distribution.

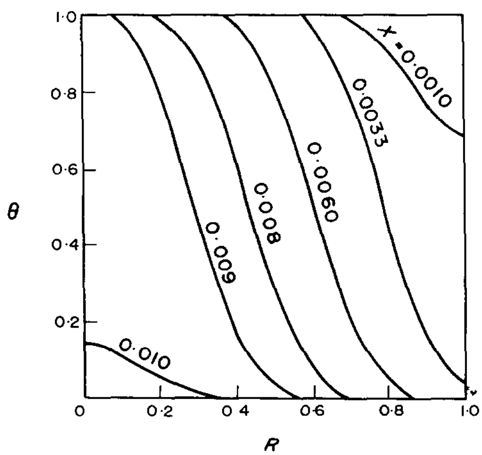

FIG. 5. Radial temperature distributions as a function of dimensionless position X at a dimensionless time $\tau = 0{\cdot}1$ for a Péclét number Pe of 1000, an adiabatic wall and the step function initial temperature distribution.

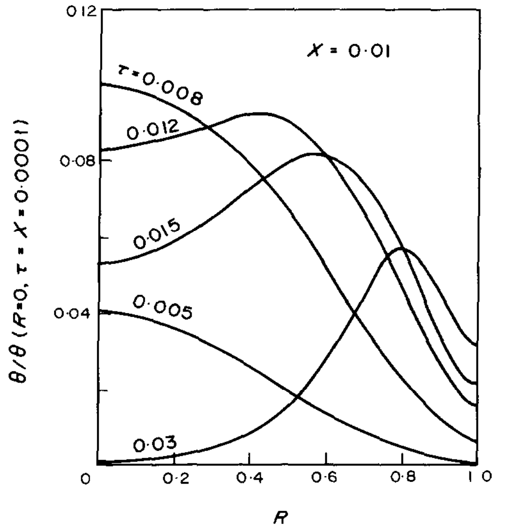

FIG. 6. Radial temperature distribution as a function of dimensionless position $X = 0{\cdot}01$ for a Péclét number Pe 1000, an adiabatic wall and an impulse form of initial temperature distribution.

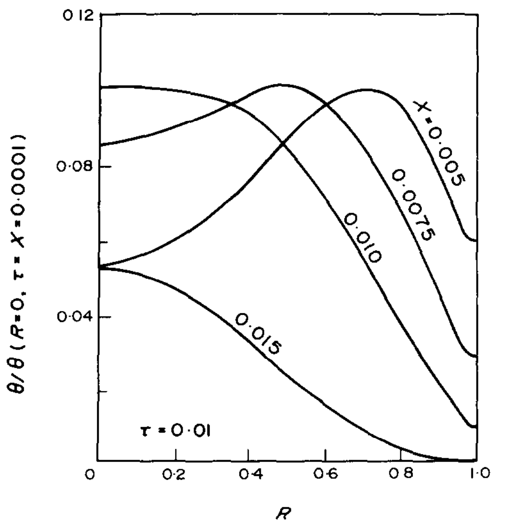

FIG. 7. Radial temperature distribution as a function of dimensionless position X at a dimensionless time $\tau = 0{\cdot}01$ for a Péclét number Pe 1000, an adiabatic wall and an impulse form of initial temperature distribution.

and 11 for the impulse initial condition and the adiabatic wall or the constant temperature wall. Figure 12 illustrates the value of

$$\sum_{n=2}^{N} \frac{2}{\alpha_n^2 - \alpha_1^2} (x_n^2 + y_n^2)$$

$\bar{k}a$ for pulsating flow. The numeral result shows that the remainder of the series,

$$\sum_{n=2}^{N} \frac{2}{\alpha_n^2 - \alpha_1^2} (x_n^2 + y_n^2),$$

from the seventh term is less than 10^{-7}. When $\bar{k}a$ is less than 0·7 the figure shows that the value of $4E_1$ is equal to $\frac{1}{6144}$ which is exactly the same as that was obtained by Aris [17]. For high frequencies it is seen that the effects of flow

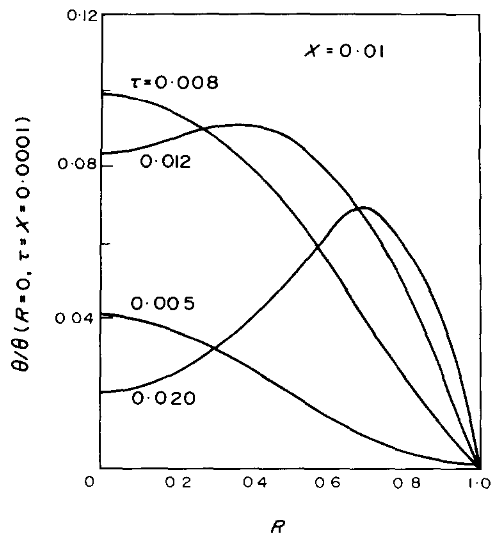

FIG. 8. Radial temperature distribution as a function of dimensionless time τ at the dimensionless position $X = 0{\cdot}01$ for a Péclét number Pe 1000 a constant wall temperature $\theta = 0$ and an impulse form of initial temperature distribution.

oscillation drops off rapidly. This method of analysis could not be used directly in the general case of time varying tube flows since the coefficient E_1 and hence the diffusion coefficient would be time dependent. The present analysis considers only the average value of E_1 over a given cycle. Alabastro and Hellums [20] have taken the perturbation approach of Lighthill for oscillating boundary layers to study diffusion with oscillating tube flows. Such a method would be restricted to thin diffusion boundary layers for steady flows with a small superimposed oscillation. In the present study harmonically oscillating flows were considered where it is shown that the effects of oscillation on diffusion is small at low frequencies and negligible at high frequencies.

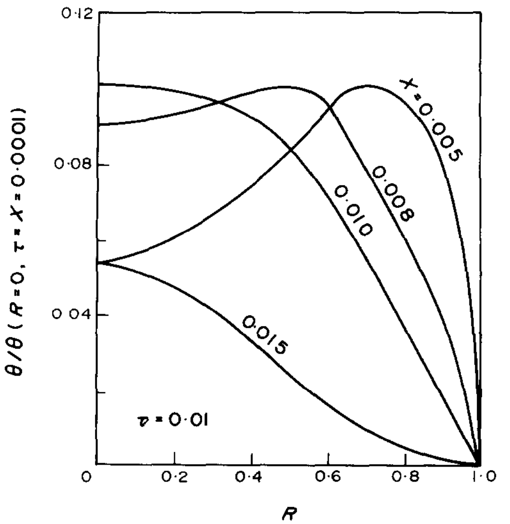

FIG. 9. Radial temperature distribution as a function of dimensionless position X at the dimensionless time $\tau = 0{\cdot}01$ for a Péclét number Pe 1000, a constant wall and an impulse form of initial temperature distribution.

The works done by Taylor [6], Aris [8], Philip [9] and Gill [10] all give methods of finding the apparent diffusion coefficient; with this information the mean concentration or temperature may be predicted from the pure diffusion model. But these methods can not give the information about the local concentration or temperature, or the amount of diffusion through the wall. It has been shown by Tseng and Besant [12] that the agreement between the numerical results of Ananthakrischnan *et al.* [14] and the present theoretical method of analysis is not exact.

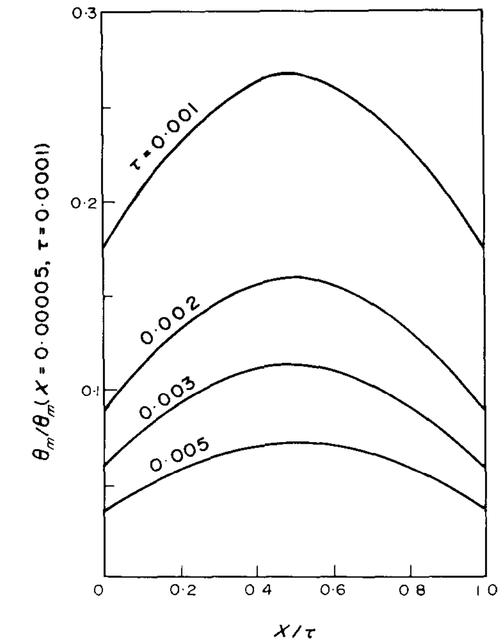

FIG. 10. The variation of the dimensionless mean temperature ratio with X/τ for $Pe = 10\,000$, an adiabatic wall and an impulse form of initial temperature distribution.

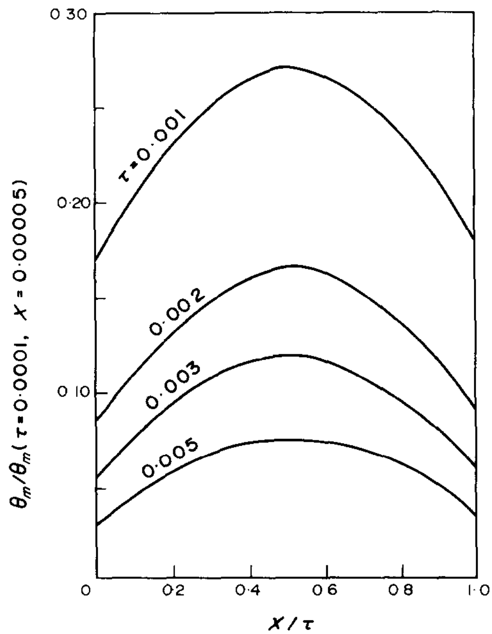

FIG. 11. The variation of the dimensionless mean temperature ratio with X/τ for $Pe = 10\,000$, an isothermal wall $\theta = 0$ and an impulse form of initial temperature distribution.

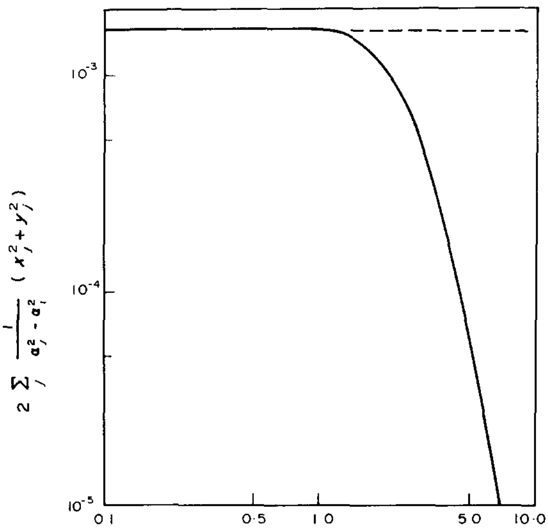

FIG. 12. The values of $4E_1$ vs. $\bar{k}a$.

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TRANSFERT THERMIQUE OU MASSIQUE TRANSITOIRE POUR UN ÉCOULEMENT LAMINAIRE ÉTABLI DANS UN TUBE

Résumé—Le problème du transfert thermique ou massique transitoire pour un écoulement laminaire établi dans un tube a été résolu exactement. Les solutions donnant la réponse à des fonctions impulsion et échelon sont considérées séparément pour des conditions aux limites soit de température ou de concentration pariétale constante, soit de paroi adiabatique. Les solutions permettent le calcul des températures ou des concentrations locales aussi bien que des flux thermiques ou massiques. On présente les résultats typiques de l'évolution, au cours du mouvement, des températures locales et du nombre de Nusselt.

INSTATIONÄRER WÄRME- UND STOFFÜBERGANG IN VOLL AUSGEBILDETEN LAMINAREN ROHRSTRÖMUNGEN

Zusammenfassung—Das Problem des instationären Wärme- oder Stoffübergangs in einer voll ausgebildeten laminaren Rohrströmung ist exakt gelöst worden. Die Lösungen für Eingangsimpuls und Eingangssprung werden getrannt betrachtet für die Randbedingungen konstanter Wandtemperatur bzw, -konzentration und für die adiabate Wand. Die Lösung gestattet die Berechnung der örtlichen Temperaturen bzw. Konzentrationen sowie des Wärme- bzw. Massenstroms. Es werden typische Ergebnisse angegeben für die stromabwärts maßgebenden örtlichen Temperaturen und Nusseltzahlen.

НЕСТАЦИОНАРНЫЙ ТЕПЛО-И МАССООБМЕН В ПОЛНОСТЬЮ РАЗВИТЫХ ЛАМИНАРНЫХ ТЕЧЕНИЯХ В ТРУБАХ

Аннотация—Решена задача о нестационарном тепло-и массообмене в полностью развитых ламинарных течениях в трубах. Решения для характеристики ступенчатой и импульсной функции рассматривались отдельно для постоянной температуры стенки и концентрации, а также для адиабатической стенки. Решение позволяет произвести расчет местных температур или концентраций, а также потоков тепла и массы. Представлены типичные результаты для локальной температуры и числа Нуссельта вниз по течению.